Lec 4: 01/28/2010

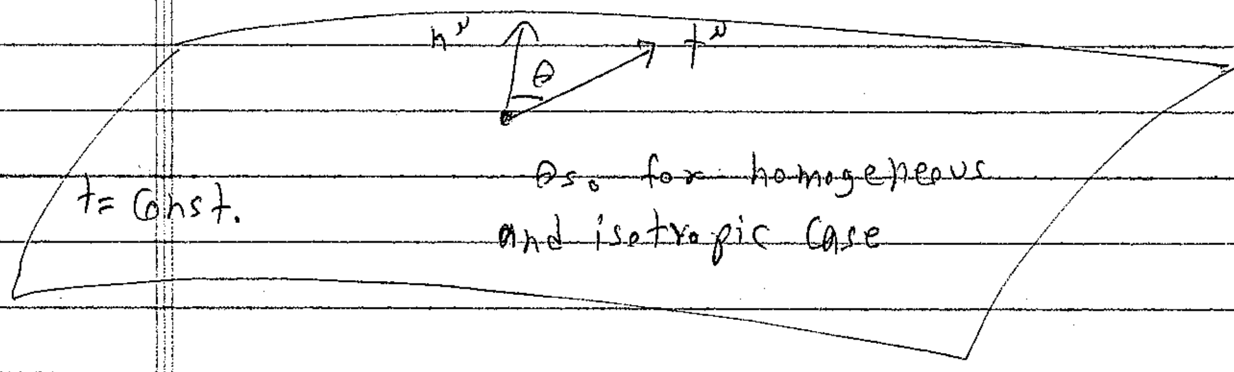
Friedmann Equations:

Now we want to describe the evolution of a homogeneous and isotropic universe according to Einstein theory of general relativity.

Initially we have a hypersurface (in 4 dimensions) that is homogeneous and isotropic. Intuitively, we expect that the hypersurface remains homogeneous and isotropic through its evolution. The key is to choose a time parameter to foliate the spacetime such that we consider hypersurfaces at a given time.

A global time function "t" is defined and the "time flow" is given by the 4-vector t^ν such that $t^\nu \nabla_\nu t = 1$.

In general, t^ν is not normal to the initial hypersurface but for a homogeneous and isotropic case we this to be the case.



"t" is called the cosmic time; observers on the initial hypersurface synchronize their clocks to some initial value. The time measured by all the clocks will be the same henceforth. Note that because of the homogeneity and isotropy, the world line of each observer who is at a fixed coordinate will be a geodesic.

The spacetime metric is given by:

$$ds^2 = -\underset{\substack{\downarrow \\ c=1 \text{ (natural units)}}}{dt^2} + a^2_{(t)}\left[dr^2 + \begin{Bmatrix} \sin^2 r \\ r^2 \\ \sinh^2 r \end{Bmatrix} (d\theta^2 + \sin^2\theta \, d\phi^2)\right]$$

Here only $a_{(t)}$ can change in time. Note that the g_{00} can also be a function of time, but it can be absorbed by a redefinition of "t".

Now the question is to find the equation(s) governing the dynamics of $a_{(t)}$. In general relativity the geometry is connected to the content. One can consider the universe as a collection of particles that form a thermodynamic system. In general, the content will be like a fluid, which is specified by its energy density "ρ", pressure "p" and stress. However, for a homogeneous and isotropic field only ρ and p will be relevant. They are related to each other through the equation of state:

$p = \omega \rho$ (natural units used, i.e. $c=1$)

Once "ω" is specified, the "perfect fluid" will be known. Two important examples of a perfect fluid that we encounter frequently in this course are:

1- Matter: generally used for "non-relativistic" particles for which 3-momentum $|\vec{p}|$ is much smaller than mass "m". In this case we have $\underline{\underline{\omega = 0}}$.

2- Radiation: generally used for "relativistic" particles for which $|\vec{p}| \gg m$. In this case we have $\underline{\underline{\omega = \frac{1}{3}}}$.

For a single component fluid, the 1st Friedmann equation is (given without derivation):

differentiation with respect to time

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \left[\frac{1}{3}\Lambda\right]$$

Here $H \equiv \frac{\dot{a}}{a}$ is the Hubble expansion rate, $K = +1; 0; -1$ for a closed; flat; open universe, and Λ is the so called cosmological constant. For the time being, we ignore the last term on the right-hand side.

This equation is not sufficient to find the scale factor $a_{(t)}$ as a function of time. Another equation can be found by using the 1st law of thermodynamics for the perfect fluid filling the universe:

internal energy $\to$ volume

$$d\overset{\epsilon}{U} = -p\,dV + T\,dS$$

$\to$ entropy, $\to$ temperature

For the universe, the expansion is adiabatic in that there is no heat reservoir outside the universe. This implies that $dS = 0$, and hence:

$$dU = -p\,dV \Rightarrow d(\rho V) = -p\,dV \Rightarrow \dot{\rho}V + \rho\dot{V} = -p\dot{V}$$

$$\Rightarrow \dot{\rho} = -\frac{\dot{V}}{V}(p+\rho)$$

But $V \propto a^3_{(t)}$, which results in $\frac{\dot{V}}{V} = 3\frac{\dot{a}}{a} = 3H$.

Thus:

$$\boxed{\dot{\rho} = -3H(p+\rho) = -3H(1+\omega)\rho}$$

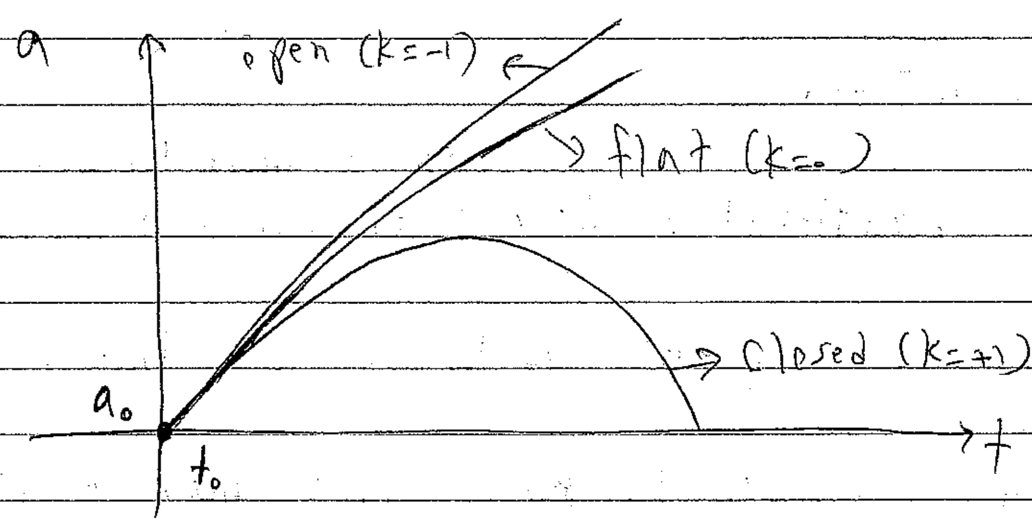
The second Friedmann equation can be obtained by taking the time derivative of both side of the 1st Friedmann equation, and use the above equation for $\dot{\rho}$. Eventually, we have the two Friedmann equations:

$$\begin{cases} \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \left[\frac{1}{3}\Lambda\right] \\ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p+\rho) + \left[\frac{1}{3}\Lambda\right] \end{cases}$$

For a multi-component fluid, we have $\rho = \sum_i \rho_i$ and $p = \sum_i p_i$ on the right-hand side of these equations (i summed over different components).

As mentioned, we ignore the cosmological constant for the moment. Lets point out some of the general properties of FRW universes:

- For $k=0$ the expansion stops only when $\rho=0$, which happens as $a \to \infty$. For $k=-1$, $\dot{a} \neq 0$ at all times. While, for $k=+1$ (and in a matter or radiation dominated universe) $\dot{a}=0$ for finite time (and finite a). In a closed universe, expansion stops and goes to contraction. One therefore has the following diagram for $a(t)$ as a function of time:



Note that the 1st Friedmann equation includes $\dot{a}^2$. Hence we can have an expanding universe ($\dot{a}>0$) or a contracting one ($\dot{a}<0$). Here we focus on the expanding solution for which there is observational evidence. In the closed case, there is transition to the contracting phase as mentioned.

— For $\rho+3p>0$ (weak energy condition) the expansion always decelerate. This is evident in the diagram on the previous page where $a(t)$ is concave in all the three ($k=+1, 0, -1$) cases. Also note that "k" does not explicitly appear in the 2nd Friedman equation.

Observational evidence (as we have for our universe) for an accelerated expansion therefore implies violation of the weak energy condition, i.e. $\rho+3p<0$. Note that in general relativity $\rho+3p$ represents gravitating energy not just ρ. As we expect $\rho>0$, then accelerated expansion requires negative pressure $p<-\frac{\rho}{3}$.

— At short times, the three cases (flat, closed, open) are not distinguishable. One has to wait long enough for the difference to become visible. Note that all we need to solve Friedmann equations are the scale factor a_0 and energy density ρ_0 at an initial time t and the equation of state.

Equivalently, to find out about the geometry of our universe from observations, we need to look at sufficiently large distances (hence sufficiently long back in time). This is intuitively understandable since S^3 and H^3 locally look like R^3.

We now find solutions of the Friedmann equations for a matter- and radiation-dominated flat universe.

(1) Matter-dominated Case: $\omega = 0$.

$\dot{\rho} = -3H(p+\rho) = -3H\rho \Rightarrow \rho = \rho_0 \left(\frac{a_0}{a}\right)^3 \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 \left(\frac{a_0}{a}\right)^3$

$$\Rightarrow \boxed{a = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3}}}$$

$$H \equiv \frac{\dot a}{a} = \frac{2}{3t}$$

(2) Radiation dominated case: $w = \frac{1}{3}$.

$\dot\rho = -3H(p+\rho) \Rightarrow \dot\rho = -4H\rho \Rightarrow \rho = \rho_0 \left(\frac{a}{a_0}\right)^4 \Rightarrow \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 \left(\frac{a}{a\right.$

$$\Rightarrow \boxed{a = a_0 \left(\frac{t}{t_0}\right)^{\frac{1}{2}}}$$

$$H \equiv \frac{\dot a}{a} = \frac{1}{2t}$$

In both cases we see that $\ddot a < 0$ (as expected, since $\rho + 3p > 0$).

Note that extrapolating back in time leads to $a \to 0$ at $t = 0$. The spacetime curvature (related to H) thus blows at $t = 0$. This signals the existence of an initial singularity where the known laws of physics (including Einstein's general relativity) break down.

We cannot use the Friedmann equations and their solutions at the singularity. It is commonly believed that a consistent theory of quantum gravity has the resolution to the singularity. Nevertheless, Friedmann equations and their solutions can be reliably used shortly after $t=0$ (say $t > t_P$, where $t_P \simeq 5.4\times10^{-44}$ s is the Planck time).

The initial singularity ($t=0$) is called the "big bang"